

# Commutativity and Wecken properties for fixed points on surfaces and 3-manifolds

Boju Jiang \*

*Carleton University, Ottawa, Ont., Canada; and Peking University, Beijing, China*

Received 2 November 1992

## *Abstract*

Jiang, B., Commutativity and Wecken properties for fixed points on surfaces and 3-manifolds, *Topology and its Application* 53 (1993) 221–228.

For a pair of maps  $\varphi: M \rightarrow P$  and  $\psi: P \rightarrow M$  between compact surfaces, the minimum number of fixed points in the homotopy class of  $\varphi \circ \psi$  may differ from that of  $\psi \circ \varphi$ . We give a sufficient condition for them to be the same, improving a recent result of M.R. Kelly. It is then applied to show that for every surface of negative Euler characteristic, the difference between the minimum number of fixed points and the Nielsen number can be arbitrarily large. The corresponding question for boundary-preserving self-maps of orientable 3-manifolds is also discussed.

**Keywords:** Surface maps; Fixed points; Minimum number of fixed points; Commutativity property; Wecken property; Boundary-Wecken property; Nielsen fixed point theory.

*AMS (MOS) Subj. Class.:* Primary 55M20; secondary 57N05, 57N10.

## 1. Introduction

A basic problem in fixed point theory is to find the minimum number of fixed points for a homotopy class of maps. That is, given a map  $f: X \rightarrow X$  of a connected compact polyhedron  $X$  into itself, to find  $MF[f] := \text{Min}\{\#\text{Fix } g \mid g \simeq f: X \rightarrow X\}$ , where  $\text{Fix } g$  denotes the set of fixed points of  $g$ . For manifolds of dimension  $n \neq 2$ ,  $MF[f]$  is equal to the Nielsen number  $N(f)$  of  $f$ , as first proved by Wecken [11, III]. The Nielsen number has the commutativity property: if  $\varphi: X \rightarrow Y$  and  $\psi: Y \rightarrow X$  are maps between connected compact polyhedra, then  $N(\psi \circ \varphi) =$

*Correspondence to:* Professor B. Jiang, Department of Mathematics, Peking University, Beijing 100871, China.

\* Partially supported by NSERC Grant A7579 and an NSFC grant.

$N(\varphi \circ \psi)$  (see e.g. [5, p. 20]). Hence if  $\varphi$  and  $\psi$  are maps between  $n$ -manifolds,  $n \neq 2$ , we have the commutativity property  $MF[\psi \circ \varphi] = MF[\varphi \circ \psi]$  for the invariant  $MF[f]$ .

For surfaces,  $MF[f]$  is no longer equal to  $N(f)$  in general, so it has become an object of study. The above commutativity property for  $MF[f]$  also breaks down for surface maps, see Example 1 below and [8, Example 1.2]. The first purpose of this paper is to prove the following restricted version of the commutativity property:

**Theorem 1.** *Let  $\varphi: M \rightarrow P$  and  $i: P \rightarrow M$  be maps between compact surfaces (closed or with boundary), where  $i: P \rightarrow M$  is an embedding that induces an injective homomorphism  $i_*: \pi_1(P) \rightarrow \pi_1(M)$ . Let  $f := i \circ \varphi: M \rightarrow M$  and  $\tilde{f} := \varphi \circ i: P \rightarrow P$ , then*

$$MF[f] = MF[\tilde{f}].$$

This kind of result first appeared in [8, Theorem 1.1] under the assumption that  $M$  has a boundary and that the embedding  $i$  induces a monomorphism on the 1-dimensional homology group. For surface embeddings,  $H_1$ -injectivity is known to imply  $\pi_1$ -injectivity. But the converse is not true, as shown by Example 2. Thus even in the case where  $M$  has a boundary, our Theorem 1 is an improvement. Its proof, adapted from [6, II, §4], is short and elementary, in contrast to the geometric arguments of [8].

As an application of Theorem 1, we discuss the difference  $MF[f] - N(f)$ . In [2], a compact connected manifold  $M$  is said to be *Wecken* if  $MF[f] = N(f)$  for all maps  $f: M \rightarrow M$  and *totally non-Wecken* if the difference  $MF[f] - N(f)$  is unbounded for self-maps  $f$  on  $M$ . For surfaces it is known [6] that  $M$  is Wecken if and only if its Euler characteristic is nonnegative.

**Theorem 2.** *Suppose  $M$  is a compact surface with Euler characteristic  $\chi(M) < 0$ . Then there is a sequence of maps  $\{f_n\}_{n>0}: M \rightarrow M$  such that  $MF[f_n] - N(f_n) \rightarrow \infty$ . Hence, compact surfaces are either Wecken or totally non-Wecken.*

This result is to be expected since Zhang [12] showed that multi-punctured disks (i.e., disks with more than one hole) are totally non-Wecken. Kelly [9, Theorem 1.1] was able to prove it for surfaces with nonempty boundary. Our proof works for all surfaces with  $\chi < 0$ . Theorem 1 is used to reduce the problem to the pants (i.e., the twice punctured disk).

The peculiar fixed point behavior of surface maps also affects the boundary-preserving maps of 3-dimensional manifolds. For a map  $f: X, A \rightarrow X, A$  of a compact polyhedral pair, define the minimum number  $MF[f; X, A] := \text{Min}\{\#\text{Fix } g \mid g \simeq f: X, A \rightarrow X, A\}$ . Schirmer [10] introduced the relative Nielsen number  $N(f; X, A)$  which is a lower bound for  $MF[f; X, A]$ . When  $M$  is a compact manifold with nonempty boundary  $\partial M$  and  $f: M, \partial M \rightarrow M, \partial M$  is a boundary-preserving map, we

shall simply write  $MF_{\partial}[f] := MF[f; M, \partial M]$  and  $N_{\partial}(f) := N(f; M, \partial M)$ . Schirmer [10] proved that for manifolds of dimension  $n \geq 4$ ,  $MF_{\partial}[f] = N_{\partial}(f)$ .

In [2], a compact connected manifold  $M$  with nonempty boundary is said to be *boundary-Wecken* if  $MF_{\partial}[f] = N_{\partial}(f)$  for all boundary-preserving maps  $f: M, \partial M \rightarrow M, \partial M$  and *totally nonboundary-Wecken* if the difference  $MF_{\partial}[f] - N_{\partial}(f)$  is unbounded for boundary-preserving self-maps  $f$  on  $M$ . Interesting results concerning the boundary-Wecken property of surfaces have been obtained in [2,9]. As another application of Theorem 1, we obtain the following result concerning 3-manifolds.

**Theorem 3.** *Suppose  $W$  is a compact orientable 3-manifold that has a component  $M$  of the boundary  $\partial W$  with Euler characteristic  $\chi(M) < 0$ . Then there is a sequence of maps  $\{g_n\}_{n>0}: W, \partial W \rightarrow W, \partial W$  such that  $MF_{\partial}[g_n] - N_{\partial}(g_n) \rightarrow \infty$ .*

*Hence, a compact orientable 3-manifold  $W$  with boundary is boundary-Wecken if and only if all its boundary components are Wecken (i.e., all have  $\chi \geq 0$ ); otherwise it is totally nonboundary-Wecken.*

It would be interesting to know if the same conclusion holds also for nonorientable 3-manifolds. One would expect that any 3-manifold is boundary-Wecken if and only if all its boundary components are Wecken.

## 2. Proof of Theorem 1

We shall write  $\partial M$  for the boundary of a surface  $M$ , and  $M^{\circ} = M \setminus \partial M$  for the interior. Up to an isotopy of the embedding  $i$  we may assume that the image  $i(P)$  is a subpolyhedron of  $M$  lying in  $M^{\circ}$ . We also assume that the embedding  $i$  is base point preserving so we shall omit the base point in our notation. By a homotopy of a map  $g: X \rightarrow Y$  we mean a continuous family of maps  $\{g_t\}_{t \in I}: X \rightarrow Y$  with  $g_0 = g$ .

**Proof of Theorem 1.** The inequality  $MF[f] \leq MF[\tilde{f}]$  is easy. Suppose  $\{\tilde{f}_t\}$  is an arbitrary homotopy of  $\tilde{f}: P \rightarrow P$ . Since  $\varphi: M \rightarrow P$  is an extension of  $\tilde{f} \circ i^{-1}: i(P) \rightarrow P$  to  $M$ , by the homotopy extension property of the polyhedral pair  $(M, i(P))$  there exists a homotopy  $\{\varphi_t\}$  of  $\varphi: M \rightarrow P$  extending  $\{\tilde{f}_t \circ i^{-1}\}$  to  $M$ . Then  $\{f_t\} := \{i \circ \varphi_t\}$  is a homotopy of  $f: M \rightarrow M$ . We have a commutative diagram

$$\begin{array}{ccccc} i(P) & \xrightarrow{\text{incl.}} & M & \xrightarrow{f_t} & M \\ \uparrow i & & \downarrow \varphi_t & & \uparrow i \\ P & \xrightarrow{\tilde{f}_t} & P & \xlongequal{\quad} & P. \end{array}$$

Hence  $i$  sends  $\text{Fix } \tilde{f}_t$  homeomorphically onto  $\text{Fix } f_t$ . This proves the inequality.

To prove the other inequality  $MF[f] \geq MF[\tilde{f}]$ , let  $\{f_t\}$  be an arbitrary homotopy of  $f: M \rightarrow M$ . It suffices to construct a homotopy  $\{\tilde{f}_t\}$  of  $\tilde{f}: P \rightarrow P$  such that  $\tilde{f}_t$  and  $f_t$  have the same number of fixed points.

(A) By attaching a collar to  $\partial M$  if  $\partial M \neq \emptyset$ , we can assume  $\bigcup_{t \in I} f_t(M) \subset M^\circ$ . More explicitly, we may replace  $M$  with  $M^c := M \cup_{\partial M} (\partial M \times I)$ , and replace  $\{f_t\}$  with  $\{f_t^c\} := \{i^c \circ f_t \circ r^c\}$ , where  $i^c: M \rightarrow M^c$  is the inclusion,  $r^c: M^c \rightarrow M$  is the retraction. Then  $f^c := f_0^c$  still satisfies our assumption, and  $\text{Fix } f_t^c = \text{Fix } f_t$ , while  $\bigcup_{t \in I} f_t^c(M^c)$  are now in the interior of  $M^c$ .

(B) Let  $p: \tilde{M} \rightarrow M$  be the covering space of  $M$  such that  $i_* \pi_1(P) = p_* \pi_1(\tilde{M})$ . Then the embedding  $i: P \rightarrow M^\circ$  can be lifted to an embedding  $i': P \rightarrow \tilde{M}^\circ$  which induces an isomorphism of the fundamental group.

The map  $f = i \circ \varphi: M \rightarrow M^\circ$  lifts to the map  $f' := i' \circ \varphi: \tilde{M} \rightarrow \tilde{M}^\circ$ . By the homotopy lifting property of covering spaces, the homotopy  $\{f_t\}: M \rightarrow M^\circ$  of  $f$  can be lifted to a homotopy  $\{f'_t\}: \tilde{M} \rightarrow \tilde{M}^\circ$  of  $f'$ . Then  $\{\tilde{f}_t\} := \{f'_t \circ p\}: \tilde{M} \rightarrow \tilde{M}^\circ$  is a lifting of  $\{f_t\}$ , i.e.,  $p \circ \tilde{f}_t = f_t \circ p$ .

(C) It is well known that an open surface with finitely generated fundamental group is the interior of a compact surface (see [1, §44D]). So  $\tilde{M}^\circ$  can be regarded as the interior of some compact surface  $\tilde{Q}$ , and the inclusion  $\tilde{P} := i'(P) \subset \tilde{Q}$  induces an isomorphism on the fundamental group. But this can happen only if  $\tilde{Q}$  is obtained from  $\tilde{P}$  by attaching a collar to  $\partial \tilde{P}$ . Thus  $\tilde{M}^\circ$  can be regarded as  $\tilde{P} \cup_{\partial \tilde{P}} \partial \tilde{P} \times [0, 1)$ , where  $[0, 1)$  stands for the half-open unit interval.

(D) Define the height function  $h: \tilde{M}^\circ \rightarrow [0, 1)$  to be 0 on  $\tilde{P}$  and to be the projection onto the second factor on  $\partial \tilde{P} \times [0, 1)$ . For  $t \in I$ , let  $m(t)$  be the maximum height on the compact subset  $\tilde{f}_t(\tilde{M}) = f'_t(\tilde{M}) \subset \tilde{M}^\circ$ . Clearly  $m: I \rightarrow [0, 1)$  is continuous. Let  $\{j_t\}_{t \in I}: \tilde{P} \rightarrow \tilde{M}^\circ$  be a continuous family of embeddings such that  $j_0$  is the inclusion and  $j_t(\tilde{P}) = \tilde{P} \cup_{\partial \tilde{P}} \partial \tilde{P} \times [0, m(t)]$ . Then we have  $j_t(\tilde{P}) \supset \tilde{f}_t(\tilde{M})$  for all  $t$ .

(E) Define a homotopy  $\{\tilde{f}_t\}: P \rightarrow P$  by

$$\tilde{f}_t(x) = i'^{-1} \circ j_t^{-1} \circ \tilde{f}_t \circ j_t \circ i'(x).$$

It is well defined because of the properties of  $\{j_t\}$ . We have

$$\tilde{f}_0 = i'^{-1} \circ \tilde{f}_0 \circ i' = i'^{-1} \circ (i' \circ \varphi \circ p) \circ i' = \varphi \circ i = \tilde{f}.$$

So  $\{\tilde{f}_t\}$  is a homotopy of  $\tilde{f}$ . We have a commutative diagram

$$\begin{array}{ccccccc} P & \xrightarrow{i'} & \tilde{P} & \xrightarrow{j_t} & \tilde{M} & \xrightarrow{p} & M \\ \tilde{f}_t \downarrow & & & & \downarrow \tilde{f}_t & & \downarrow f_t \\ P & \xrightarrow{i'} & \tilde{P} & \xrightarrow{j_t} & \tilde{M} & \xrightarrow{p} & M. \end{array}$$

Now  $j_t \circ i'$  maps  $\text{Fix } \tilde{f}_t$  homeomorphically onto  $\text{Fix } \tilde{f}_t$  because  $j_t \circ i'(P) \supset \tilde{f}_t(\tilde{M})$ , and  $p$  maps  $\text{Fix } \tilde{f}_t$  homeomorphically onto  $\text{Fix } f_t$  because  $\tilde{f}_t$  factors through  $M$  via  $f'_t$ . Hence  $\tilde{f}_t$  and  $f_t$  have the same number of fixed points, as desired.  $\square$

**Example 1.** Let  $A$  be the annulus and  $P$  be the pants. In [6, II, §3] we constructed a map  $\phi: S^1 \rightarrow P$  such that for any map  $\psi: P \rightarrow S^1$  we have  $MF[\phi \circ \psi] > 0$ . Let  $m = 1$  in the choice of  $\psi$  so that the Lefschetz number  $L(\phi \circ \psi) = 0$ . Regard the annulus  $A$  as a fattened  $S^1$  and naturally modify the maps into  $\phi: A \rightarrow P$  and  $\psi: P \rightarrow A$  by composing the old ones with the retraction  $A \rightarrow S^1$  and the inclusion  $S^1 \rightarrow A$  respectively. Then  $MF[\phi \circ \psi] > 0$  but on the annulus  $MF[\psi \circ \phi] = N(\psi \circ \phi) = 0$ .

In this example  $\phi_*: \pi_1(A) \rightarrow \pi_1(P)$  is injective and  $\phi$  can be made an immersion. This shows the importance of the embedding assumption in Theorem 1, the  $\pi_1$ -injectivity assumption alone is not sufficient. Of course the embedding assumption by itself is also inadequate, as shown by [8, Example 1.2].

**Example 2.** The  $\pi_1$ -injectivity assumption in Theorem 1 is weaker than the  $H_1$ -injectivity condition of [8]: Let  $M$  be the punctured torus and let  $A$  be an annulus around the puncture. Then for the homomorphisms induced by the inclusion,  $i_*: \pi_1(A) \rightarrow \pi_1(M)$  is injective but  $i_*: H_1(A) \rightarrow H_1(M)$  is zero. More interesting examples can be built on this one. For example  $M$  can be the twice punctured torus and  $P$  can be the pants around the punctures.

### 3. Proof of Theorem 2

In this section  $P$  always stands for the pants, that is, it is the sphere with three holes. Choose an orientation of  $P$  and let its boundary curves  $C_0, C_1$  and  $C_2$  be oriented accordingly. When a map  $\psi: P \rightarrow S^1$  is given, let  $d_k$  be the degree of the restriction  $\psi|_{C_k}: C_k \rightarrow S^1$ ,  $k = 0, 1, 2$ ; the degrees satisfy the equality  $d_0 + d_1 + d_2 = 0$ . By switching the labels of the curves and changing the orientation of  $P$  if necessary, we can always assume  $d_1 \geq d_2 \geq 0$ . When  $P$  is presented in the plane as a disk with two holes, we draw  $C_0$  as the counterclockwise outer boundary and  $C_1$  and  $C_2$  as the clockwise inner boundaries. Choose a base point in the interior  $P^\circ$ . Let  $\rho_1, \rho_2$  be two simple closed curves in  $P^\circ$ , enclosing  $C_1$  and  $C_2$  respectively, and meeting at the base point to form a figure “8”. Orient them clockwise. The loops  $\rho_1, \rho_2$  form a free basis of the fundamental group  $\pi_1(P)$ .

**Proof of Theorem 2.** (A) Observe that since  $\chi(M) < 0$ , every component of  $\partial M$  is a boundary curve of some  $\pi_1$ -injective pants  $P$  in  $M$ . We claim that if  $C$  is a 2-sided simple closed curve in  $M^\circ$  such that  $\pi_1(C)$  injects into  $\pi_1(M)$ , then  $C$  is also a boundary curve of some  $\pi_1$ -injective pants  $P$  in  $M$ .

To see this, cut  $M$  along  $C$  to obtain a surface  $M'$  with  $\chi(M') = \chi(M) < 0$ . There are two possibilities. (1)  $M'$  is connected. Since  $C$  is 2-sided in  $M$  (with regular neighborhood an annulus rather than a Möbius band),  $M$  is obtained from  $M'$  by gluing together two components  $C'_1, C'_2$  of  $\partial M'$ . Such a gluing always induces a monomorphism  $\pi_1(M') \rightarrow \pi_1(M)$ . Now  $C'_1$  as a component of  $\partial M'$  is a

boundary curve of a  $\pi_1$ -injective  $P$  in  $M'$ . By pushing it away from  $C'_2$  if necessary, we can assume  $P$  does not intersect  $C'_2$ . This  $P$ , when regarded as embedded in  $M$ , is thus  $\pi_1$ -injective and has  $C$  as a boundary curve, as required. (2)  $M'$  has two pieces  $M'_1$  and  $M'_2$ , without loss we may assume  $\chi(M'_1) < 0$ .  $M$  is now obtained from  $M'_1$  by pasting  $M'_2$  to it along closed curves  $C'_1 \subset \partial M'_1$  and  $C'_2 \subset \partial M'_2$ . The attached  $M'_2$  is not a disk because  $\pi_1(C)$  injects into  $\pi_1(M)$ . Hence  $\pi_1(M'_1)$  injects into  $\pi_1(M)$ . In  $M'_1$  we can find a  $\pi_1$ -injective  $P$  with  $C'_1$  as a boundary curve, so it is  $\pi_1$ -injective in  $M$  as well. Our claim is proved.

(B) It is clear that there always exists a map  $\psi : M \rightarrow S^1$  such that its restriction to some simple closed curve  $C$  in  $M^\circ$  has nonzero degree. Thus  $\psi|_C$  induces a monomorphism  $\pi_1(C) \rightarrow \pi_1(S^1)$ , so  $\pi_1(C)$  injects into  $\pi_1(M)$ . We can assume that  $C$  is 2-sided in  $M$ , because otherwise we can replace it by the boundary  $C'$  of its regular neighborhood (a Möbius band) which is certainly 2-sided. Hence, it follows from (A) that we can find embedded pants  $P$  in  $M$  such that the inclusion  $i : P \rightarrow M$  induces a monomorphism of the fundamental group, and the degrees  $d_k$  of  $\psi \circ i|_{C_k}$  satisfy  $d_1 > 0$  and  $d_2 \geq 0$ .

(C) Now choose a sequence

$$\omega_n := (\rho_1 \rho_2 \rho_1^{-1} \rho_2^{-1})^n, \quad n \geq 1$$

in  $\pi_1(P)$ . Let  $\phi_n : S^1 \rightarrow P$  be a map representing  $\omega_n$ .

Define  $f_n := i \circ \phi_n \circ \psi : M \rightarrow M$  and  $\tilde{f}_n := \phi_n \circ \psi \circ i : P \rightarrow P$ . Let  $f'_n := \psi \circ i \circ \phi_n : S^1 \rightarrow S^1$ .

By the commutativity property of the Nielsen number,  $N(f_n) = N(\tilde{f}_n) = N(f'_n)$ . It is clear that  $f'_n$  has degree 0. Hence  $N(f_n) = 1$ .

(D) Theorem 1 tells us  $MF[f_n] = MF[\tilde{f}_n]$ . The map  $\tilde{f}_n : P \rightarrow P$  induces

$$\tilde{f}_n * : \pi_1(P) \rightarrow \pi_1(P), \quad \rho_1 \mapsto \omega_n^{d_1}, \quad \rho_2 \mapsto \omega_n^{d_2}.$$

To estimate  $MF[\tilde{f}_n]$ , there are two different approaches available. We can use the more algebraic technique of [12] to prove that  $MF[\tilde{f}_n] \geq (d_1 + d_2)n - 1$ . Or we may apply the algorithm of [7] to get  $MF[\tilde{f}_n] = 2(d_1 + d_2)n - 1$ . Therefore  $MF[f_n] - N(f_n) = MF[\tilde{f}_n] - 1 \rightarrow \infty$  when  $n \rightarrow \infty$ .  $\square$

It is interesting to note that in a surface  $M$  with  $\chi(M) = -1$  (other than  $P$  itself) it is impossible to find an  $H_1$ -injective embedding of  $P$ . It is the weaker  $\pi_1$ -injectivity assumption of Theorem 1 that enables us to give a unified proof for all surfaces of interest.

#### 4. Proof of Theorem 3

In this section,  $W$  is always a compact 3-manifold with boundary. We keep the notational convention in Section 3 concerning the pants  $P$ .

**Proof of Theorem 3.** It is easy to see that if all boundary components of  $W$  are Wecken, then  $W$  is boundary-Wecken [10, Theorem 6.2]. So the second conclusion of the theorem follows from the first.

For the proof of the former, we shall construct the required sequence of maps  $\{g_n\}_{n>0}: W, \partial W \rightarrow W, \partial W$  with the additional property that  $g_n(W) \subset M$ .

(A) The inclusion  $j: M \rightarrow W$  induces a nontrivial homomorphism  $j^* \neq 0: H^1(W) \rightarrow H^1(M)$  on the integral cohomology. Hence there exists  $h \in H^1(W)$  such that  $j^*(h) \neq 0$ .

To see this, first consider the case where  $\partial W$  is connected, i.e.,  $M = \partial W$ . Suppose to the contrary  $j^* = 0: H^1(W) \rightarrow H^1(\partial W)$ . Then from the exact sequence of the reduced cohomology

$$H^1(\partial W) \xleftarrow{j^*} H^1(W) \longleftarrow H^1(W, \partial W) \longleftarrow \tilde{H}^0(\partial W)$$

we see  $H^1(W, \partial W) \cong H^1(W)$ . On the other hand, since  $W$  is orientable, by the Lefschetz duality theorem we know  $H^1(W, \partial W) \cong H_2(W)$ . Thus, the Betti numbers  $\beta_0(W) = 1$ ,  $\beta_1(W) = \beta_2(W)$  and  $\beta_3(W) = 0$ . So  $\chi(W) = 1$  and  $\chi(\partial W) = 2\chi(W) = 2$ , contradicting our hypothesis on  $\chi(M)$ .

Now consider the case where there are components of  $\partial W$  other than  $M$ . By gluing a handlebody (or a 3-disk) to  $W$  along each such component, we obtain a larger orientable 3-manifold  $W'$  with  $\partial W' = M$ . The inclusion  $j': M \rightarrow W'$  factors through  $j$ . From the previous case we see  $j'^* \neq 0: H^1(W') \rightarrow H^1(M)$ , hence  $j^* \neq 0: H^1(W) \rightarrow H^1(M)$ .

(B) Homotopy theory tells us (cf. [4, §II.7]) that there is a bijection between  $H^1(W)$  and the set  $[W, S^1]$  of homotopy classes of maps  $W \rightarrow S^1$ . An element  $a \in H^1(W)$  corresponds to the homotopy class of a map  $\alpha: W \rightarrow S^1$  if and only if for every oriented simple closed curve  $C$  in  $W$ , the value of  $a$  on  $C$  equals the degree of the restriction  $\alpha|_C: C \rightarrow S^1$ . So the cohomology class  $h \in H^1(W)$  in (A) provides us a map  $\theta: W \rightarrow S^1$  whose restriction on some simple closed curve  $C$  in  $M$  has nonzero degree.

(C) As in (B) in Section 3, by virtue of (A) in Section 3 we can find a  $\pi_1$ -injective embedding  $i$  of  $P$  into  $M$  such that the degrees  $d_k$  of  $\theta \circ i|_{C_k}$  satisfy  $d_1 > 0$  and  $d_2 \geq 0$ .

(D) Let  $\phi_n: S^1 \rightarrow P$  be as in (C) in Section 3. We obtain a cyclic diagram of maps:

$$\begin{array}{ccc} P & \xleftarrow{\phi_n} & S^1 \\ i \downarrow & & \uparrow \theta \\ M & \xrightarrow{j} & W \end{array}$$

Define  $g_n := j \circ i \circ \phi_n \circ \theta: W, \partial W \rightarrow W, \partial W$ . Define  $f_n := i \circ \phi_n \circ \theta \circ j: M \rightarrow M$  and  $\tilde{f}_n := \phi_n \circ \theta \circ j \circ i: P \rightarrow P$ . Let  $f'_n := \theta \circ j \circ i \circ \phi_n: S^1 \rightarrow S^1$ .

It is easy to see from the definition of  $N_\partial(g_n)$  that  $N_\partial(g_n) = N(f_n)$ . By the commutativity property of the Nielsen number,  $N(f_n) = N(f'_n) = 1$ , because  $f'_n$  has degree 0. Hence  $N_\partial(g_n) = 1$ .

(E) We claim that  $MF_{\partial}[g_n] = MF[f_n]$ . Indeed, the inequality  $MF_{\partial}[g_n] \geq MF[f_n]$  is obvious. To see that  $MF_{\partial}[g_n] \leq MF[f_n]$ , suppose  $\{h_t\}: M \rightarrow M$  is a homotopy of  $f_n$ . By the homotopy extension property of the pair  $(W, M)$ ,  $\{h_t\}$  extends to a homotopy  $\{h'_t\}: W \rightarrow M$  of the map  $i \circ \phi_n \circ \theta$ . Now  $\{k_t := j \circ h'_t\}: W, \partial W \rightarrow W, \partial W$  is a homotopy of  $g_n$  which coincides with  $\{h_t\}$  on  $M$ . Since the image of  $k_t$  is entirely in  $M$ , it has the same fixed points as  $h_t$ .

(F) Exactly as in (D) in Section 3, we have  $MF[f_n] = MF[\tilde{f}_n] \geq (d_1 + d_2)n - 1$ . Thus  $MF_{\partial}[g_n] - N_{\partial}(g_n) = MF[\tilde{f}_n] - 1 \rightarrow \infty$  when  $n \rightarrow \infty$ .  $\square$

It is an open question whether this theorem holds also for nonorientable 3-manifolds. The orientability is essential in step (A) of the above proof.

If the nonorientable  $W$  satisfies the condition  $j^* \neq 0: H^1(W) \rightarrow H^1(M)$  (to bypass step (A)), then the above construction still works. This condition has several equivalent forms. One is that some homotopically nontrivial map  $M \rightarrow S^1$  can be extended to a map  $W \rightarrow S^1$ . Another form is that some simple closed curve  $C$  in  $M$  represents a nontorsion element in  $H_1(W)$ . However, this homological condition is not always satisfied. See [3, pp. 64–65] for counterexamples.

### Acknowledgement

The author is in debt to Helga Schirmer for helpful discussions. Theorem 2 as an application of Theorem 1 was suggested to the author by Robert Brown.

### References

- [1] L.V. Ahlfors and L. Sario, *Riemann Surfaces* (Princeton University Press, Princeton, NJ, 1960).
- [2] R.F. Brown and B.J. Sanderson, Fixed points of boundary-preserving maps of surfaces, *Pacific J. Math.* 158 (1993) 243–264.
- [3] J. Hempel, *3-Manifolds*, *Annals of Mathematics Studies* 86 (Princeton University Press, Princeton, NJ, 1976).
- [4] S.-T. Hu, *Homotopy Theory* (Academic Press, New York, 1959).
- [5] B. Jiang, *Lectures on Nielsen Fixed Point Theory*, *Contemporary Mathematics* 14 (American Mathematical Society, Providence, RI, 1983).
- [6] B. Jiang, Fixed points and braids, *Invent. Math.* 75 (1984) 69–74; II, *Math. Ann.* 272 (1985) 249–256.
- [7] M.R. Kelly, Minimizing the number of fixed points for self-maps of compact surfaces, *Pacific J. Math.* 126 (1987) 81–123.
- [8] M.R. Kelly, Minimizing the cardinality of the fixed point set for self-maps of surfaces with boundary, *Michigan Math. J.* 39 (1992) 201–217.
- [9] M.R. Kelly, The relative Nielsen number and boundary-preserving surface maps, Preprint.
- [10] H. Schirmer, A relative Nielsen number, *Pacific J. Math.* 122 (1986) 459–473.
- [11] F. Wecken, Fixpunktklassen I, *Math. Ann.* 117 (1941) 659–671; II, *Math. Ann.* 118 (1942) 216–234; III, *Math. Ann.* 118 (1942) 544–577.
- [12] X.-G. Zhang, The least number of fixed points can be arbitrary larger than the Nielsen number, *Acta Sci. Natur. Univ. Pekinensis* (3) (1986) 15–25.